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Optimality of the Centered Form for Polynomials

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R. E. Moore ("Interval Analysis," Prentice–Hall, Englewood Cliffs, N. J., 1966) introduced the centered form for approximating the range $\bar{f}(X)$ of a rational function f over X, where X is a real interval. H. Ratschek (*SIAM J. Numer. Anal.* 17(1980), 656–662), introduced centered forms of higher order. These are, in general, better approximations than the centered form. If f is a polynomial p, however, then the centered form and the centered form of higher order lead to the same approximation. This distinguished behavior of polynomials is investigated and it is shown that the centered form is the best possible approximation of $\bar{p}(X)$, if the centered form is compared with all approximations of $\bar{p}(X)$ that include $\bar{p}(X)$ and depend on the same data of p as those needed in constructing the centered form.

1. INTRODUCTION

Let p be a real polynomial. One is frequently required to compute an approximation of the range of p over a real interval X, namely, $\bar{p}(X) = \{p(x): x \in X\}$. A natural tool for obtaining such an approximation is the centered form of p computed in interval arithmetic.

Interval arithmetic was defined by Moore [3], who also introduced the centered form. Here we give a short discussion of the reasons for using interval arithmetic as well as some of its elementary properties.

Present-day computers employ an arithmetic commonly called fixed length floating point arithmetic. In this arithmetic real numbers are approximated by a subset of the real numbers called the machine representable numbers (in short, machine numbers). These are of the form βb^{α}, where b is the base and

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α and β both have a fixed number of digits throughout the calculation procedure. Any calculation done gives a result of that form which is usually the closest machine representable value (in short, machine value) to the actual value for that operation. There are, therefore, two sources of error, for both real valued data and intermediate results must be approximated by machine numbers.

Interval arithmetic provides a tool for automatically estimating and controlling the effect of these errors. Instead of approximating a real value x by a machine value, a pair of machine values representing an interval is found such that x lies between these values; that is, such that x lies within the interval. This leads naturally to the development of an arithmetic for intervals.

The real number $\frac{1}{3}$ cannot be represented by a machine number. It can, however, be enclosed in the interval $A = [0.33, 0.34]$ (assuming two digit β's and α's with $b = 10$). If we now want to multiply $\frac{1}{3}$ by a quantity c which we know lies in $B = [-0.01, 0.02]$, then we seek the smallest interval X which:

(a) contains $c/3$,

(b) does not depend on c and $\frac{1}{3}$, and

(c) depends only on the intervals A and B.

The result of employing these constraints on an arithmetic for intervals is given below.

Let $I(R)$ be the set of real compact intervals (only these are considered usually). Then operations on $I(R)$ satisfying (a), (b) and (c) are defined by

$$A * B = \{a * b : a \in A, b \in B\} \qquad \text{for} \quad A, B \in I(R). \tag{1}$$

The symbol * stands for $+$, $-$, $\cdot$, and $/$, and A/B is only defined if $0 \notin B$. Since (1) is useless in practical calculations, the following formulas which are equivalent to (1) are preferred (see Moore [3]):

$$\begin{aligned}
[a, b] + [c, d] &= [a + c, b + d], \\
[a, b] - [c, d] &= [a - d, b - c], \\
[a, b][c, d] &= [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)], \\
[a, b]/[c, d] &= [a, b][1/d, 1/c] \qquad \text{if} \quad 0 \notin [c, d].
\end{aligned}$$

We write a and $-A$ instead of $[a, a]$ and $(-1)A$ if $a \in R$ (set of reals) and $A \in I(R)$. Hence expressions like $a + A$, $a - A$, A/a, etc., are meaningful.

Powers of intervals may be defined in two different ways. The first version is called the *simple* version and it is defined by

$$A^0 = 1 \qquad \text{and} \qquad A^n = A \cdot \cdots \cdot A \ (n \text{ times}) \qquad \text{if} \quad n \geqslant 1.$$

If A is symmetric, that is, $A = [-a, a]$ then $A^n = [-a^n, a^n]$ if $n \geqslant 1$. The second version is called the *extended* version and it is defined by

$$A^n = \{a^n : a \in A\} \qquad \text{for} \quad n \geqslant 0.$$

If $A = [-a, a]$ then $A^0 = 1$ and

$$\begin{aligned} A^n &= [-a^n, a^n], && \text{if } n \text{ is odd} \\ &= [0, a^n], && \text{if } n \text{ is even, } n \neq 0. \end{aligned}$$

For example, if $c, d \in A$, then A^2 in the simple version is the smallest interval that contains cd, and A^2 in the extended version is the smallest interval that contains c^2.

The following procedure is usually followed in interval analysis (especially if A is symmetric): The theoretical investigations are first done in the simple version. Then, if suitable, the practical calculations are done in the extended version because the resulting intervals are then smaller. It is therefore appropriate to consider both versions in the sequel.

The *width* or *diameter* of an interval $A = [a, b]$ is denoted by $d(A) = b - a$. If $c \in A$ then $d(A)$ is a measure of how well c is approximated by A.

Now let p again be a polynomial and $X \in I(R)$. The *centered form for p over X* is defined as the interval

$$P(X) = p(c) + \sum_{i=1}^{n} p^{(i)}(c)(X - c)^i / i!,$$

where c is the midpoint of X and n the degree of p; cf. [1, 3].

The centered form has interesting practical and theoretical properties. For example, the inclusion

$$\bar{p}(X) \subset P(X)$$

always holds. Practically, it was investigated in [3], where it was argued that it was a *good* method for actually approximating the range of a polynomial. As a further example, in [1, 2] the centered form was shown to converge to the range of the function with a *quadratic convergence* on the width of the interval X.

The centered form furthermore offers remarkable computational advantages because all the intervals $(X - c)^i$ are symmetric in the simple version. Other suggestions for reducing the width of the approximation use Bernstein polynomials; cf. [5, 6].

The present paper now demonstrates the *reason* for these good results. That is, we prove that the centered form is optimal under all approximations of the form $\sum_{i=0}^{m} b_i H^i$ that include the range of p over X where the coefficients b_i depend on the *data*, $p(c)$, $p'(c)$,..., $p^{(n)}(c)$, and $H = X - c$ is the *centering* of X.

Our demand that the admitted approximations shall depend on the data mentioned above results from the fact that the centered form also depends on these data; and a *reasonable* comparison between various kinds of approximations is certainly only then possible if for all these approximations the same information, in our case the same data, is available. (Please notice that this concept of dependence of the calculation on certain restricted information, and not on all of f, is very realistic and occurs practically, for example, in physical observations, measurements, storage limitations of computers, etc.) Furthermore, we demand that the approximation can be calculated by a computer. For this reason we exclude transcendental operations; that is, the approximation shall be a rational expression of $p(c)$,..., $p^{(n)}(c)$ and H. In particular, the coefficients b_i are rational functions. Therefore the approximations can be seen as algorithms, where the parameters (inputs) are $p(c)$,..., $p^{(n)}(c)$, and H.

2. Approximations and Optimality

In this section the definitions are given that are necessary for a precise treatment of the ideas developed in Section 1.

Let $\mathbb{P}$ be the class of real polynomials of degree at most n. The *data* of a polynomial over the interval X will be defined as the operators

$$W_i : \mathbb{P} \to R \text{ with } W_i(p) = p^{(i)}(c) \qquad \text{for} \quad i = 0,..., n.$$

Using this definition the centered form of $p \in \mathbb{P}$ over X may be written as

$$P(X) = \sum_{i=0}^{n} W_i(p) H^i/i!.$$

Let $\beta : R^{n+1} \times I(R) \to I(R)$ be an interval valued polynomial over $I(R)$ of the form

$$\beta(v_0,..., v_n, Y) = \sum_{i=0}^{m} b_i(v_0,..., v_n) Y^i, \tag{2}$$

where the b_i's are rational functions in $v_0,..., v_n$.

Then we call β an *including approximation of the range of the polynomials of* $\mathbb{P}$ *with respect to the data* $W_0,..., W_n$ (abbreviated, an *approximation for* $\mathbb{P}$), if the *inclusion condition*

$$\beta(W_0(p),..., W_n(p), H) \supset \bar{p}(X) \tag{3}$$

holds for all $p \in \mathbb{P}$ and all $X \in I(R)$ with $H = X - c$. The background of the demand that (3) shall hold for all $X \in I(R)$ is the intent that we want to obtain an algorithm with H (and the data) as inputs. Otherwise, it would be necessary to find an algorithm for each X.

To avoid long expressions we will write v instead of $(v_0, \dots, v_n)$ such that v is a variable over R^{n+1} and we write W_p instead of $(W_0(p), \dots, W_n(p))$.

Comparing representation (2) with condition (3) for $X = [c, c]$ we get the value of the absolute coefficient of each approximation β for $\mathbb{P}$, that is,

$$b_0(v) = v_0. \tag{4}$$

In order to give reasonable criteria for an approximation for $\mathbb{P}$ to be optimal, we have to decide how to compare the approximations. There are two practical possibilities which are commonly used:

DEFINITION 1. An approximation β for $\mathbb{P}$ is called *optimal with respect to inclusion* [respectively, *optimal with respect to the width*] if for each approximation γ for $\mathbb{P}$

$$\gamma(W_p, H) \subset \beta(W_p, H) \qquad [\text{respectively, } d(\gamma(W_p, H)) \leqslant d(\beta(W_p, H))]$$

holds for all $p \in \mathbb{P}$ and all symmetrical $H \in I(R)$ only then if $\gamma = \beta$.

Thus, an optimal approximation is a minimal element in the partially ordered set that consists of the class of all approximations for $\mathbb{P}$, where the order relation is either given by the inclusion or by the "less or equal" relation with respect to the width of the approximating intervals. Definition 1 is independent of a special interval X for the same reason that the definition of an approximation for $\mathbb{P}$ is.

A connection between the two kinds of optimal approximations is given below:

LEMMA. *If the approximation β is optimal with respect to the width then β is optimal with respect to inclusion.*

Proof. Clear, because $\gamma(W_p, H) \subset \beta(W_p, H)$ implies $d(\gamma(W_p, H)) \leqslant d(\beta(W_p, H))$ for any approximation γ. Q.E.D.

3. OPTIMALITY OF THE CENTERED FORM

In this section it will be shown that the centered form is an optimal approximation for $\mathbb{P}$ with respect to inclusion. The optimality with respect to the width will only be shown in the case of simple calculation of powers; cf.

Section 1. As in the previous section, c will always denote the midpoint of the interval X and $H = X - c = [-z, z]$.

The following theorem holds for both kinds of power calculation.

THEOREM. *The centered form is optimal with recpect to inclusion.*

Proof. Let $\beta(v, Y) = \sum_{i=0}^{m} b_i(v)\, Y^i$ be any approximation for $\mathbb{P}$ such that

$$\beta(W_p, H) \subset P(X) \tag{5}$$

for all $p \in \mathbb{P}$ and all $X \in I(R)$. Then we have to show that $\beta(W_p, H) = P(X)$ for all p and X, or equivalently, that

$$\beta(v, H) = \sum_{i=0}^{n} v_i H^i / i!. \tag{6}$$

We divide the proof into two steps.

Step 1. Let $p \in \mathbb{P}$ be such that

$$W_k(p) > 0 \qquad \text{for} \quad k = 1, \ldots, n. \tag{7}$$

Then we will show that $\beta(W_p, H) = P(X)$ for all intervals X (respectively that $b_i(W_p)\, H^i = w_i H^i$ $(i = 1, \ldots, n)$, where $w_i = W_i(p)/i!$ $(i = 0, \ldots, n)$).

We develop p in a Taylor series around c,

$$p(x) = \sum_{i=0}^{n} w_i (x - c)^i.$$

We see that p is monotonically increasing in $[c, \infty)$. We define

$$X_0 = [c, c + z] \qquad \text{and} \qquad H_0 = X_0 - c = [0, z].$$

From the monotonicity, $p(X_0) = \bar{p}(X_0)$ follows; cf. [3]. From the assumptions and the inclusion $X_0 \subset X$ we get

$$P(X) \supset \beta(W_p, H) \supset \bar{p}(X) \supset \bar{p}(X_0) = p(X_0). \tag{8}$$

Denoting the right endpoint of an interval A by $\mathrm{re}(A)$ and writing down $P(X)$ and $p(X_0)$ explicitly one can see that $\mathrm{re}(P(X)) = \mathrm{re}(p(X_0))$. Therefore, by (8),

$$\mathrm{re}(P(X)) = \mathrm{re}(\beta(W_p, H)). \tag{9}$$

Now we have to distinguish two cases corresponding to the two kinds of powers:

(a) *Simple version.* The symmetry of the powers H^i for $i \geqslant 1$ implies that

$$P(X) = p(c) + (\mathrm{re}(P(X)) - p(c))[-1, 1]$$

and

$$\beta(W_p, H) = b_0(W_p) + (\mathrm{re}(\beta(W_p, H)) - b_0(W_p))[-1, 1].$$

From (9) and $b_0(W_p) = w_0 = p(c)$, cf. (4), we conclude that

$$\beta(W_p, H) = P(X).$$

(b) *Extended version.* Equality (9) means that for all $z \geqslant 0$

$$\sum_{i=0}^{n} w_i z^i = w_o + \sum_{i=1,\, i\,\mathrm{odd}}^{m} |b_i(W_p)|\, z^i + \sum_{i=2,\, i\,\mathrm{even}}^{m} \max\{0, b_i(W_p)\}\, z^i.$$

By comparing coefficients it follows that $m = n$ and

$$w_i = |b_i(W_p)| \qquad \text{for odd } i,$$
$$w_i = b_i(W_p) \qquad \text{for even } i \geqslant 2,$$

which proves Step 1.

Step 2. We only handle the case of extended power calculation. Then the Theorem is proved by showing that the functions b_i which are rational in v are of the following form:

$$\begin{aligned} &b_i(v) = v_i/i! \text{ or } b_i(v) = -v_i/i! && \text{if } i \geqslant 2 \text{ is even, } i \leqslant n, \\ &b_i(v) = v_i/i! && \text{if } i \geqslant n \text{ is odd,} \\ &b_i(v) = 0 && \text{if } i > n. \end{aligned} \tag{10}$$

Now, Eqs. (10) hold for all v with $v_1,..., v_n > 0$. One can see this in the following way:

Let v be fixed with $v_1,..., v_n > 0$. Then there exists a $p \in \mathbb{P}$ with $W_i(p) = v_i$ for $i = 1,..., n$. It follows that $\beta(W_p, H) = P(X)$ by Step 1. Writing down this equation (with $H = [-z, z]$ as variable) and comparing coefficients we just obtain (10) for this fixed v. If i is even, the choice of the sign in (10) depends on v. Because (10) must hold for any such v and since b_i is a rational function,

$$b_i(v) = v_i/i! \text{ for all such } v \quad \text{or} \quad b_i(v) = -v_i/! \text{ for all such } v.$$

Now, $b_i(v)$ and $v_i/v!$ are two rational functions which are identical on the set of all $v \in R^{n+1}$ with $v_1, \ldots, v_n > 0$. Hence they are identical for all $v \in R^{n+1}$.

We proceed analogously for odd indices i and also for the simple case of power calculation. Q.E.D.

In the following Corollary only the simple version for computing powers is permitted:

COROLLARY. *The centered form for* $\mathbb{P}$ *is an optimal approximation with respect to the width.*

Proof. Let β be an approximation for $\mathbb{P}$ of form (2) such that $d(\beta(W_p, H)) \geqslant d(P(X))$ for all $p \in \mathbb{P}$ and all $X \in I(R)$. Because the midpoints of $\beta(W_p, H)$ and $P(X)$ are equal, we conclude $\beta(W_p, H) \subset p(X)$, and $\beta(W_p, H) = P(X)$ by the Theorem. Q.E.D.

Remark 1. The proof of the Corollary cannot be transferred to the extended version of power calculation, because the proof is based on the identity of the two midpoints. This identity cannot be guaranteed in the extended version because the powers H^i are not necessarily symmetric any more.

Remark 2. The centered forms of higher order which were introduced in [4] do not lead to better approximations for polynomials than the centered form, as is the case for rational functions. This follows directly from the Theorem because the centered form of higher order for polynomials is also of the form $\sum_{i=0}^{m} b_i H^i$, where the coefficients b_i depend rationally on the admitted data.

4. SUPPLEMENTARY REMARKS

In Section 3, we saw that the centered form is optimal in the class of all approximations for $\mathbb{P}$.

In Section 2 care was taken of the precise definition of the approximation for $\mathbb{P}$. That is, only such expressions are admitted as approximations that are rational functions in the data and H. It is now shown that if other operations leading to non-rational expressions are allowed, then a better inclusion may be obtained.

EXAMPLE. The following algorithm lead to a better inclusion than the centered form:

If $[W_i(p) \geqslant 0$ for $i = 1, 2, \dots, n$ *and* $W_i(p) = 0$ for $i = 4, 6, 8, \dots, n$ *and* $W_2(p) \leqslant 2W_1(p) \leqslant W_3(p)]$, *then*

$$\beta(W_p, H) = \sum_{i=0,\, i \text{ even}}^{n} W_i(p)\, z^i/i! + \sum_{i=1,\, i \text{ odd}}^{n} W_i(p)[-z^i, z^i]/i!$$

else

$$\beta(W_p, H) = W_0(p) + \sum_{i=1}^{n} W_i(p)\, H^i/i!.$$

This algorithm is the centered form for the *else* statement, otherwise it is an improvement on the centered form. The first clause says that p is monotonically increasing in $[c, c+z]$ whereas the two following clauses say that p is monotonically increasing in $[c-z, c]$. Clearly, if all the conditions hold, then

$$\bar{p}(x) = [p(c-z), p(c+z)]$$

as just given by the formula of the *then* statement.

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